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FDM for the integral-differential equation of the hyperbolic type

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Abstract

In this paper, the second order of accuracy difference scheme approximately solving the initial value problem for an integral-differential equation of the hyperbolic type in a Hilbert space H is presented. The stability estimates for the solution of this difference scheme are obtained. Theoretical results are supported by numerical examples.

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1 Introduction

We consider the initial value problem

$$\begin{cases} \frac{d^2u(t)}{dt^2} + Au(t) = \int_{-t}^{t} B(\rho)u(\rho)\,d\rho + f(t), & -1 \le t \le 1, \\ u(0) = u_0, \qquad u'(0) = u'_0, \end{cases} \tag{1}$$

for an integral-differential equation in a Hilbert space H with unbounded linear operators A and $B(t)$ in H with dense domain $D(A) \subset D(B(t))$ and

$$\left\| B(t)A^{-1} \right\|_{H\to H} \le M, \quad -1 \le t \le 1. \tag{2}$$

It is well known that various initial-boundary value problems for the integral-differential equation of the hyperbolic type with two dependent limits can be reduced to the initial value problem (1) in a Hilbert space H; see [1–3].

A function $u(t)$ is called a *solution* of the problem (1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on $[-1,1]$. The derivative at the endpoints of the segment are understood as the appropriate unilateral derivatives.

(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [-1,1]$, and the function $Au(t)$ is continuous on $[-1,1]$.

(iii) $u(t)$ satisfies the equations and the initial conditions (1).

A solution of the problem (1) defined in this manner will from now on be referred to as a solution of the problem (1) in the space $C(H) = C([-1,1],H)$ of all continuous functions $\varphi(t)$ defined on $[-1,1]$ with values in H equipped with the norm

$$\|\varphi\|_{C(H)} = \max_{-1\le t\le 1} \left\|\varphi(t)\right\|_H.$$

We consider the problem (1) under the assumption that A is a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > \delta_0 > 0$.

Theorem 1 [4] *Suppose that $u_0 \in D(A)$, $u_0' \in D(A^{1/2})$ and $f(t)$ is a continuously differentiable function on $[-1,1]$. Then there is a unique solution of the problem (1) and the stability inequalities*

$$\begin{aligned}
&\max_{-1\leq t\leq 1}\left\|\frac{d^2u(t)}{dt^2}\right\|_H + \max_{-1\leq t\leq 1}\left\|Au(t)\right\|_H \\
&\quad\leq M^*\left[\|Au_0\|_H + \left\|A^{1/2}u_0'\right\|_H + \left\|f(0)\right\|_H + \int_{-1}^{1}\left\|f'(s)\right\|_H\,ds\right]
\end{aligned} \tag{3}$$

hold, where M^* does not depend on u_0, u_0', and $f(t)$, $t \in [-1,1]$.

In [4] the first order of accuracy difference scheme

$$\begin{cases}
\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_{k+1} = \sum_{j=-k}^{k} B_j u_j \tau + f_k, \quad k = 1,\dots,N-1,\\
\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_{k-1} = -\sum_{j=k}^{-k} B_j u_j \tau + f_k, \quad k = -N+1,\dots,0,\\
\tau = \frac{1}{N}, \qquad B_k = B(t_k), \qquad f_k = f(t_k), \qquad t_k = k\tau, \quad k = -N,\dots,N,\\
u_0 = u(0), \qquad (I+\tau^2 A)\frac{u_1-u_0}{\tau} = u_0'
\end{cases} \tag{4}$$

for approximate solutions of the problem (1) was considered.

Theorem 2 [4] *Suppose that the requirements of Theorem 1 are satisfied. Then for the solution of difference scheme (4) the stability inequalities*

$$\begin{aligned}
&\max_{-N+1\leq k\leq N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H + \max_{-N\leq k\leq N}\|Au_k\|_H \\
&\quad\leq M^*\left[\|Au_0\|_H + \left\|A^{1/2}u_0'\right\|_H + \|f_0\|_H + \sum_{k=-N+1}^{N}\|f_k - f_{k-1}\|_H\right]
\end{aligned} \tag{5}$$

hold, where M^* does not depend on u_0', u_0, and f_k, $k = -N,\dots,N$.

In this paper, we consider the second order of accuracy difference scheme

$$\begin{cases}
\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1}+u_{k-1})\\
\quad = \tau\sum_{j=-k+1}^{k} B_{j-\frac{1}{2}}\left(\frac{u_j+u_{j-1}}{2}\right) + f_k, \quad k = 1,\dots,N-1,\\
\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + \frac{1}{2}Au_k + \frac{1}{4}A(u_{k+1}+u_{k-1})\\
\quad = -\tau\sum_{j=k+1}^{-k} B_{j-\frac{1}{2}}\left(\frac{u_j+u_{j-1}}{2}\right) + f_k, \quad k = -N+1,\dots,-1,\\
\tau = \frac{1}{N}, \qquad f_k = f(t_k), \qquad t_k = k\tau, \quad k = -N,\dots,N,\\
B_{k-\frac{1}{2}} = B(t_k - \frac{\tau}{2}), \quad k = -N+1,\dots,N,\\
u(0) = u_0, \qquad (I+\tau^2 A)\left(\frac{u_1-u_0}{\tau}\right) = \frac{\tau}{2}(f_0 - Au_0) + u_0',\\
(I+\tau^2 A)\left(\frac{u_0-u_{-1}}{\tau}\right) = \frac{\tau}{2}(Au_0 - f_0) + u_0',
\end{cases} \tag{6}$$

for approximate solutions of the problem (1). The paper is organized as follows. In Section 2 we obtain the stability estimates for the solution of difference scheme (6). Numerical illustrations for the simple test problem are provided in Section 3. The paper is concluded with remarks in Section 4.

2 The stability estimates for the solution of difference scheme

Theorem 3 *Suppose that the requirements of Theorem* 1 *are satisfied. Then for the solution of difference scheme* (6) *the stability inequalities*

$$\begin{aligned} &\max_{-N+1\le k\le N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H+\max_{-N+1\le k\le N}\left\|\frac{A(u_k+u_{k-1})}{2}\right\|_H \\ &\quad\le M^*\left[\|Au_0\|_H+\left\|A^{1/2}u_0'\right\|_H+\|f_0\|_H+\sum_{k=-N+1}^{N}\|f_k-f_{k-1}\|_H\right] \end{aligned} \tag{7}$$

hold, where M^* does not depend on u_0', u_0, and f_k, $k=-N,\dots,N$.

Proof By [5], the second order of accuracy difference scheme

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=\psi_k, & k=1,\dots,N-1, \\ (I+\tau^2A)\frac{u_1-u_0}{\tau}=\frac{\tau}{2}(\psi_0-Au_0)+u_0', & u_0=u(0) \end{cases} \tag{8}$$

has the solution

$$\begin{aligned} u_1&=\left(I+\tau^2A\right)^{-1}\left[\left(I+\frac{\tau^2}{2}A\right)u_0+\tau u_0'+\frac{\tau^2}{2}\psi_0\right], \\ u_k&=\left[R^k+\frac{1}{2i}A^{-1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\right. \\ &\quad\times\left.\left\{\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau}{2}A-iA^{1/2}\left(I+\tau^2A\right)\right\}\left(I+\tau^2A\right)^{-1}\right]u_0 \\ &\quad+\frac{i}{2}A^{-1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)u_0' \\ &\quad+\frac{i}{2}A^{-1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau}{2}\psi_0 \\ &\quad-\sum_{j=1}^{k-1}\frac{\tau}{2i}A^{-1/2}\left(R^{k-j}-\tilde{R}^{k-j}\right)\psi_j, \quad k=2,\dots,N, \end{aligned}$$

where $R=(I-\frac{i\tau A^{1/2}}{2})(I+\frac{i\tau A^{1/2}}{2})^{-1}$, $\tilde{R}=(I+\frac{i\tau A^{1/2}}{2})(I-\frac{i\tau A^{1/2}}{2})^{-1}$. By putting $\psi_0=f_0$, $\psi_k=\tau\sum_{s=-k+1}^{k}B_{s-\frac{1}{2}}(\frac{u_s+u_{s-1}}{2})+f_k$, $k=1,\dots,N-1$, we obtain

$$\begin{aligned} Au_k&=\left[R^k+\frac{1}{2i}A^{-1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\right. \\ &\quad\times\left.\left\{\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau}{2}A-iA^{1/2}\left(I+\tau^2A\right)\right\}\left(I+\tau^2A\right)^{-1}\right]Au_0 \\ &\quad+\frac{i}{2}A^{1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)u_0' \\ &\quad+\frac{i}{2}A^{1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau}{2}f_0 \end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{k-1}\frac{\tau^2}{2i}A^{1/2}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)\sum_{s=-j+1}^{j}B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\\
&-\sum_{j=1}^{k-1}\frac{\tau A^{1/2}}{2i}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)f_j,\quad k=2,\dots,N. \qquad (9)
\end{aligned}
$$

Since $i\tau A^{1/2}=(I-\frac{i\tau A^{1/2}}{2})(\tilde{R}-I)=(I+\frac{i\tau A^{1/2}}{2})(I-R)$, we have

$$
\begin{aligned}
&\sum_{j=s}^{k-1}\frac{\tau A^{1/2}}{2i}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)=\frac{1}{2}\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-s}+\frac{1}{2}\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-s}-I,\\
&\sum_{j=-s+1}^{k-1}\frac{\tau A^{1/2}}{2i}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)=\frac{1}{2}\left(I+\frac{i\tau A^{1/2}}{2}\right)R^{k+s}+\frac{1}{2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k+s}-I.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\frac{\tau^2A^{1/2}}{2i}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)\sum_{s=-j+1}^{j}B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\\
&\quad=\sum_{s=1}^{k-1}\sum_{j=s}^{k-1}\frac{\tau A^{1/2}}{2i}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau\\
&\qquad+\sum_{s=-k+2}^{0}\sum_{j=-s+1}^{k-1}\frac{\tau A^{1/2}}{2i}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau\\
&\quad=\frac{1}{2}\sum_{s=1}^{k-1}\left[\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-s}+\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-s}-2I\right]B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau\\
&\qquad+\frac{1}{2}\sum_{s=-k+2}^{0}\left[\left(I+\frac{i\tau A^{1/2}}{2}\right)R^{k+s}+\left(I-\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k+s}-2I\right]B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau. \qquad (10)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\frac{\tau A^{1/2}}{2i}\bigl(R^{k-j}-\tilde{R}^{k-j}\bigr)f_j\\
&\quad=-f_{k-1}+\frac{1}{2}\left[\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-1}+\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-1}\right]f_0\\
&\qquad+\frac{1}{2}\sum_{j=1}^{k-1}\left[\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-j}+\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-j}\right](f_j-f_{j-1}). \qquad (11)
\end{aligned}
$$

Putting (10)-(11) in (9), we get

$$
\begin{aligned}
Au_k=\Biggl[&R^k+\frac{1}{2i}A^{-1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\bigl(R^k-\tilde{R}^k\bigr)\\
&\times\left\{\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau}{2}A-iA^{1/2}\bigl(I+\tau^2A\bigr)\right\}\bigl(I+\tau^2A\bigr)^{-1}\Biggr]Au_0
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{2}A^{1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)u_0'\\
&+\frac{i}{2}A^{1/2}\left(I-\frac{i\tau A^{1/2}}{2}\right)\left(R^k-\tilde{R}^k\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau}{2}f_0\\
&+f_{k-1}-\frac{1}{2}\left[\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-1}+\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-1}\right]f_0\\
&-\frac{1}{2}\sum_{j=1}^{k-1}\left[\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-j}+\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-j}\right](f_j-f_{j-1})\\
&+\frac{1}{2}\sum_{s=1}^{k-1}\left[2I-\left(I-\frac{i\tau A^{1/2}}{2}\right)R^{k-s}-\left(I+\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k-s}\right]B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau\\
&+\frac{1}{2}\sum_{s=-k+2}^{0}\left[2I-\left(I+\frac{i\tau A^{1/2}}{2}\right)R^{k+s}-\left(I-\frac{i\tau A^{1/2}}{2}\right)\tilde{R}^{k+s}\right]B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau,\\
&k=2,\dots,N.
\end{aligned}
\tag{12}
$$

Using $I+R^{-1}=2(I-\frac{i\tau A^{1/2}}{2})^{-1}$ and $I+\tilde{R}^{-1}=2(I+\frac{i\tau A^{1/2}}{2})^{-1}$, from (12) we obtain

$$
\begin{aligned}
\frac{A(u_k+u_{k-1})}{2}={}&\Bigg[R^k\left(I-\frac{i\tau A^{1/2}}{2}\right)^{-1}+\frac{A^{-1/2}}{2i}\left(R^k-\tilde{R}^{k-1}\right)\\
&\times\left\{\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau A}{2}-iA^{1/2}\left(I+\tau^2A\right)\right\}\left(I+\tau^2A\right)^{-1}\Bigg]Au_0\\
&+\frac{iA^{1/2}}{2}\left(R^k-\tilde{R}^{k-1}\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)u_0'\\
&+\frac{iA^{1/2}}{2}\left(R^k-\tilde{R}^{k-1}\right)\left(I+\tau^2A\right)^{-1}\left(I+\frac{i\tau A^{1/2}}{2}\right)\frac{\tau}{2}f_0\\
&+\left(I-\frac{R^{k-1}+\tilde{R}^{k-1}}{2}\right)f_0+\sum_{j=1}^{k-1}\left(I-\frac{R^{k-j}+\tilde{R}^{k-j}}{2}\right)(f_j-f_{j-1})\\
&+\sum_{s=1}^{k-1}\left(I-\frac{R^{k-s}+\tilde{R}^{k-s}}{2}\right)B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau\\
&+\sum_{s=-k+1}^{0}\left(I-\frac{R^{k+s-1}+\tilde{R}^{k+s-1}}{2}\right)B_{s-\frac{1}{2}}\left(\frac{u_s+u_{s-1}}{2}\right)\tau,\\
&k=2,\dots,N.
\end{aligned}
\tag{13}
$$

Then, using (2) and the following estimates:

$$
\begin{aligned}
&\|R\|_{H\to H}\le 1,\qquad \|\tilde{R}\|_{H\to H}\le 1,\qquad \left\|\left(I\pm\frac{i\tau A^{1/2}}{2}\right)^{-1}\right\|_{H\to H}\le 1,\\
&\left\|\left(I\pm i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\le 1,\qquad \left\|\tau A^{1/2}\left(I\pm i\tau A^{1/2}\right)^{-1}\right\|_{H\to H}\le 1
\end{aligned}
$$

yields

$$\left\| \frac{A(u_k + u_{k-1})}{2} \right\|_H \le \frac{5}{2} \|Au_0\|_H + \left\| A^{1/2} u_0' \right\|_H + \frac{5}{2} \|f_0\|_H + 2 \sum_{j=-N+1}^{N} \|f_j - f_{j-1}\|_H + 2M\tau \sum_{j=-k+1}^{k-1} \left\| \frac{A(u_j + u_{j-1})}{2} \right\|_H, \tag{14}$$

where $k = 2, \dots, N$. Furthermore, we have

$$\frac{A(u_1 + u_0)}{2} = \left[I + \left(I + \tau^2 A \right)^{-1} \left(I + \frac{\tau^2 A}{2} \right) \right] \frac{Au_0}{2} + \frac{\tau A}{2} \left(I + \tau^2 A \right)^{-1} u_0' + \frac{\tau^2 A}{4} \left(I + \tau^2 A \right)^{-1} f_0,$$

which gives us

$$\left\| \frac{A(u_1 + u_0)}{2} \right\|_H \le \|Au_0\|_H + \frac{1}{2} \left\| A^{1/2} u_0' \right\|_H + \frac{1}{4} \|f_0\|_H. \tag{15}$$

In a similar way, one can prove that

$$\left\| \frac{A(u_k + u_{k-1})}{2} \right\|_H \le \frac{5}{2} \|Au_0\|_H + \left\| A^{1/2} u_0' \right\|_H + \frac{5}{2} \|f_0\|_H + 2 \sum_{j=-N+1}^{N} \|f_j - f_{j-1}\|_H + 2M\tau \sum_{j=k+1}^{-k-1} \left\| \frac{A(u_j + u_{j-1})}{2} \right\|_H \tag{16}$$

holds for $k = -N + 1, \dots, -1$ and

$$\left\| \frac{A(u_0 + u_{-1})}{2} \right\|_H \le \|Au_0\|_H + \frac{1}{2} \left\| A^{1/2} u_0' \right\|_H + \frac{1}{4} \|f_0\|_H. \tag{17}$$

Using (14)-(17) and the theorem about the discrete analog of a Gronwall type integral inequality with two dependent limits [4, 6], we obtain

$$\max_{-N+1 \le k \le N} \left\| \frac{A(u_k + u_{k-1})}{2} \right\|_H \le \tilde{M} \left[\|Au_0\|_H + \left\| A^{1/2} u_0' \right\|_H + \|f_0\|_H + \sum_{k=-N+1}^{N} \|f_k - f_{k-1}\|_H \right]. \tag{18}$$

Finally, using the triangle inequality in (6) we complete the proof of the estimates (7). □

3 Numerical example

We consider the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = \int_{-t}^{t} \frac{\partial^2 u(s,x)}{\partial x^2} \, ds + \left(\frac{2t^3}{3} + t^2 + 2 \right) \sin x, & -1 \le t \le 1, 0 < x < \pi, \\ u(0,x) = 0, \quad u_t(0,x) = 0, & 0 \le x \le \pi, \\ u(t,0) = 0, \quad u(t,\pi) = 0, & -1 \le t \le 1, \end{cases} \tag{19}$$

Table 1 The errors between the exact solution of the problem (19) and the numerical solutions computed by using the first order and the second order of accuracy difference schemes (20) and (21), respectively

	$N = M = 16$	$N = M = 32$	$N = M = 64$
First order of accuracy difference scheme (20)	0.0765	0.0384	0.0192
Second order of accuracy difference scheme (21)	0.0016	0.0004	0.0001

which has the exact solution $u(t,x) = t^2 \sin x$. Applying the first order of accuracy difference scheme (4) to the problem (19) yields

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} = \tau \sum_{i=-k}^{k} \frac{u_{n+1}^i-2u_n^i+u_{n-1}^i}{h^2} + (\frac{2t_k^3}{3} + t_k^2 + 2)\sin x_n, \\ \quad n = 1,\dots,M-1, k = 1,\dots,N-1, \\ \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2} = -\tau \sum_{i=k}^{-k} \frac{u_{n+1}^i-2u_n^i+u_{n-1}^i}{h^2} + (\frac{2t_k^3}{3} + t_k^2 + 2)\sin x_n, \\ \quad n = 1,\dots,M-1, k = -N+1,\dots,0, \\ \tau = 1/N, \qquad h = \pi/M, \qquad t_k = k\tau, \quad k = -N,\dots,N, \\ x_n = nh, \quad n = 0,\dots,M, \\ u_n^0 = u_n^1 = 0, \quad n = 0,\dots,M, \qquad u_0^k = u_M^k = 0, \quad k = -N,\dots,N. \end{cases} \tag{20}$$

Similarly, applying the second order of accuracy difference scheme (6) to the problem (19), we have

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2} - \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2} \\ \quad = (\frac{2t_k^3}{3} + t_k^2 + 2)\sin x_n + \tau \sum_{i=-k+1}^{k} (\frac{u_{n+1}^i-2u_n^i+u_{n-1}^i}{2h^2} + \frac{u_{n+1}^{i-1}-2u_n^{i-1}+u_{n-1}^{i-1}}{2h^2}) \\ \qquad n = 1,\dots,M-1, k = 1,\dots,N-1, \\ \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2} - \frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{2h^2} - \frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{4h^2} - \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{4h^2} \\ \quad = (\frac{2t_k^3}{3} + t_k^2 + 2)\sin x_n - \tau \sum_{i=k+1}^{-k} (\frac{u_{n+1}^i-2u_n^i+u_{n-1}^i}{2h^2} + \frac{u_{n+1}^{i-1}-2u_n^{i-1}+u_{n-1}^{i-1}}{2h^2}) \\ \qquad n = 1,\dots,M-1, k = -N+1,\dots,-1, \\ \tau = 1/N, \qquad h = \pi/M, \qquad t_k = k\tau, \quad k = -N,\dots,N, \\ x_n = nh, \quad n = 0,\dots,M, \\ u_n^0 = 0, \qquad u_n^1 = u_n^{-1} = \tau^2 \sin x_n, \quad n = 0,\dots,M, \\ u_0^k = u_M^k = 0, \quad k = -N,\dots,N. \end{cases} \tag{21}$$

The difference schemes (20) and (21) are implemented by using the Gauss Elimination Method in Matlab. The errors are computed by

$$E = \max_{\substack{-N \le k \le N \\ 0 \le n \le M}} \left| u(t_k, x_n) - u_n^k \right|,$$

where u_n^k represents the numerical solution of the difference schemes at (t_k, x_n). Table 1 shows the errors between the exact solution and the numerical solutions computed by using the first order and the second order of accuracy difference schemes (20) and (21), respectively. Table 1 is constructed using numerical solutions of the difference schemes for different values of N and M. We observe that both schemes converge with the correct order.

4 Conclusion

In this paper we have studied the second order of accuracy difference scheme approximately solving the initial value problem (1) for an integral-differential equation of the hyperbolic type in a Hilbert space H. The stability estimates for the solution of this difference scheme have been obtained. We have been able to confirm the correct order of the difference scheme by a numerical illustration for the simple test problem.

The aim of our future work is to apply high order of approximation two-step difference schemes [7–10] for an approximate solution of the initial value problem (1).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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